Applications of Double Integrals.

Density and mass

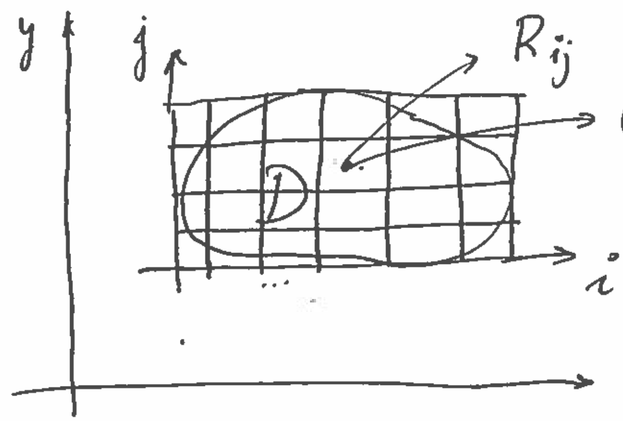
We use the function

$\rho(x,y)$: density (mass/area) at point (x,y)

in a region D. of xy-plane.

Assume ρ is a continuous function in D.

Total mass of a lamina occupying a region D in the plane with $\rho(x,y)$ as it's density at any point $(x,y) \in D$.



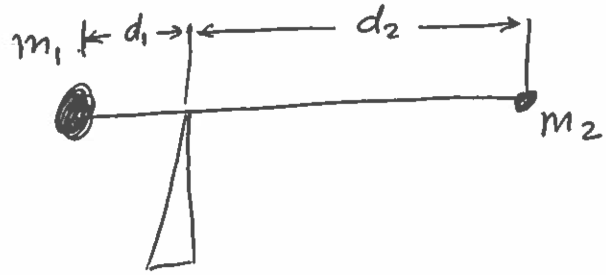
(x_{ij}^*, y_{ij}^*) a point in R_{ij}

Area $R_{ij} = \Delta A$
uniform partition.

$$\text{Total mass } D \simeq \sum_{i=1}^{k} \sum_{j=1}^{\ell} \rho(x_{ij}^*, y_{ij}^*) \Delta A$$

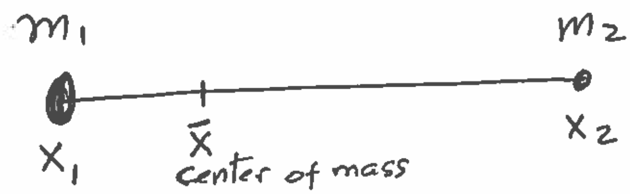
Moments and Center of Mass.

Two particles balancing along a rod



Law of the Lever

$$\boxed{m_1 d_1 = m_2 d_2}$$



$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$

$$\Rightarrow \quad m_1\bar{x} + m_2\bar{x} = m_1 x_1 + m_2 x_2$$

or
$$\boxed{\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}}$$

$m_1 x_1$ and $m_2 x_2$ moments of masses m_1 and m_2 with respect to origin

By increasing the number of rectangles and $\Delta A \to 0$ we get

$$m = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\,\Delta A$$

$$\overset{\text{by def. of integral}}{=} \iint_D \rho(x,y)\,dA$$

Electric charges and charge density

$\sigma(x,y)$: Charge density; units charge/unit Area.

For example Coulombs/m^2.

$$\text{Total charge} = Q = \iint_D \sigma(x,y)\,dA .$$

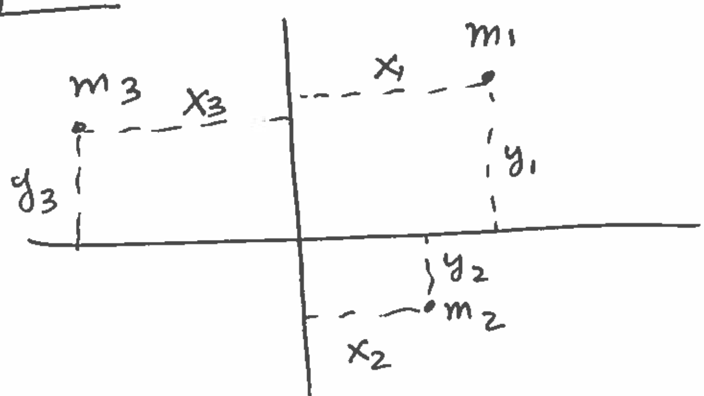
$$\boxed{\text{Center of mass} = \frac{\sum \text{moments of masses}}{\text{Total mass.}}}$$

If n particles

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i = m}$$

<u>Definition</u> $M = \sum_{i=1}^{n} m_i x_i$ <u>moment of the System about origin.</u>

<u>In the plane</u>



Moment of System about y-axis $= M_y = \sum_{i=1}^{n} m_i x_i$

Moment of System about x-axis $= M_x = \sum_{i=1}^{n} m_i y_i$

Definition

Center of mass $(\bar{x}, \bar{y})$ such that

$$\bar{x} = \frac{M_y}{m}, \qquad \bar{y} = \frac{M_x}{m}$$

Continuous distribution of particles in a region D of xy-plane

Given density $\rho(x,y)$ in D

moment of R_{ij} w.r.t. x-axis

$$\left(\rho(x_{ij}^*, y_{ij}^*)\Delta A\right) y_{ij}^*$$

For the entire lamina occupying the region D.

Moment about x-axis $= M_x = \lim\limits_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} y_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\Delta A$

$$= \iint_D y\,\rho(x,y)\,dA$$

Similarly,

Moment about y-axis $= \iint_D x\,\rho(x,y)\,dA$

Definition

Center of mass $(\bar{x}, \bar{y})$ of a lamina occupying the region D and having density function $\rho(x,y)$ are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint_D x\rho(x,y)\, dA$$

$$\bar{y} = \frac{M_x}{m} = \frac{1}{m} \iint_D y\,\rho(x,y)\, dA$$

Where $m = \iint_D \rho(x,y)\, dA$

Probability

Continuous Random Variables x.

- Values of x range over an interval of real numbers.

For example,

$5ft \le x \le 6ft.$ height of a person

Probability density function:

Every continuous random variable x has a "probability density function".

This function is such that

$$\text{Probability } a \le x \le b =$$

$$= P(a \le x \le b) = \int_a^b f(x)\,dx$$

In general, prob. density function x satisfies

① $f(x) \geq 0$

② $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$

In two Variables, we consider two random Variables. For example,

X : Height and Y: weight

Then, we define the <u>Joint Density Function</u> of x and y, $f(x,y)$

As in the 1-D case,

Probability ($a \le x \le b$ and $c \le y \le d$)

$$P(a \le x \le b,\ c \le y \le d) = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

Also, $f(x,y) \geq 0$, $(x,y) \in \mathbb{R}^2$

and $$\iint_{\mathbb{R}^2} f(x,y)\, dA = 1$$

$$\parallel$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dA$$

15.6

#27) Joint density function of x and y

$$f(x,y) = \begin{cases} Cx(1+y), & 0 \le x \le 1 \text{ and } 0 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$$

a) Find C, b) Find $P(x \le 1, y \le 1)$

c) Find $P(x+y \le 1)$

Answ: To find C, we use the property of $f(x,y)$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1.$$

Then,

$$\int_0^1 \int_0^2 Cx(1+y)\,dy\,dx =$$

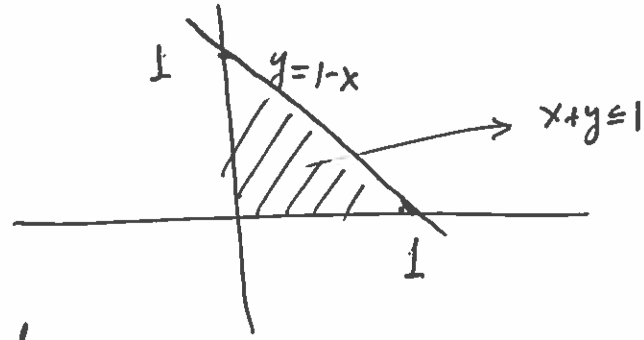
$$= \int_0^1 Cx\left(y + \frac{y^2}{2}\right)\Big|_0^2 dx = \int_0^1 Cx(4-0)\,dx = 4C\frac{1}{2} = 2C = 1$$

$$\Rightarrow \boxed{C = 1/2}$$

b) $P(x \le 1, y \le 1) = \int_0^1 \int_0^1 \frac{1}{2}x(1+y)\,dy\,dx =$

$$\int_0^1 \frac{1}{2}x\left(y + \frac{y^2}{2}\right)\Big|_0^1 dx = \frac{1}{2}\int_0^1 x\,\frac{3}{2}\,dx = \frac{3}{4}\frac{x^2}{2}\Big|_0^1 = \frac{3}{8}$$

c) $P(x+y \leq 1)$



$$\int_{0}^{1} \int_{0}^{1-x} \frac{1}{2} x(1+y)\, dy\, dx$$

$\quad x \quad\ y$

Expected Values

1-D $\quad$ Mean $= \mu = \displaystyle\int_{-\infty}^{\infty} x f(x)\, dx$

2-D

$$X\text{-mean} = \mu_1 = \iint_{\mathbb{R}^2} x f(x,y)\, dA \xrightarrow{\text{Similar}} M_y$$

$$y\text{-mean} = \mu_2 = \iint_{\mathbb{R}^2} y f(x,y)\, dA \xrightarrow{\text{similar}} M_x$$

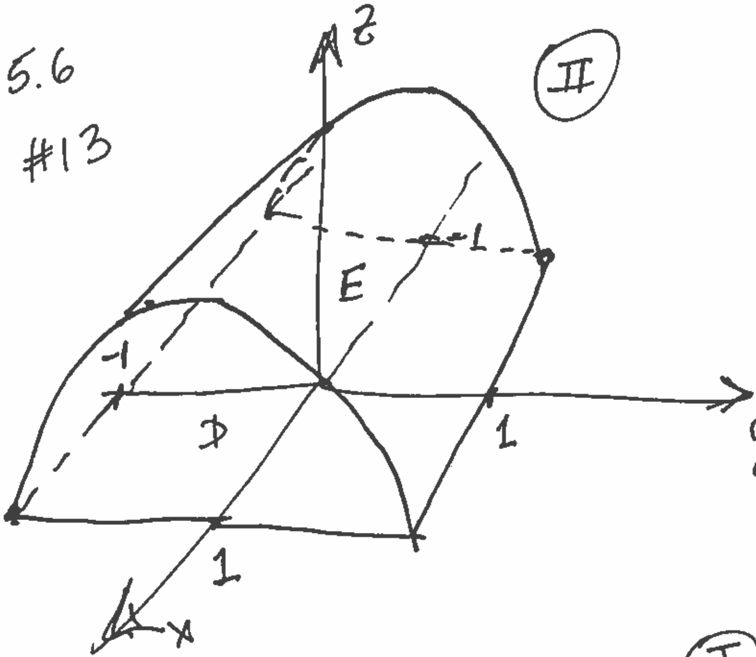
Since $\displaystyle\iint_{\mathbb{R}^2} f(x,y)\, dA = 1$

$$(X\text{-mean},\ y\text{-mean}) = (\mu_1, \mu_2) \xrightarrow{\text{Similar}} (\bar{x}, \bar{y})$$

Center of mass.

15.6

#13



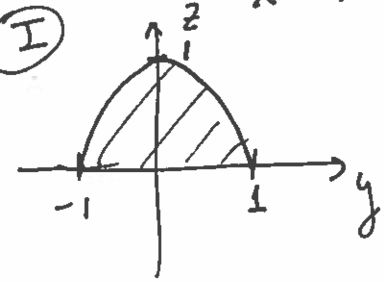
$$I = \iiint_E x^2 e^y \, dV$$

E Solid bounded by

$z = 1 - y^2$
$z = 0$
$x = 1$
$x = -1$

what are all these surfaces in the 3D-space?

I



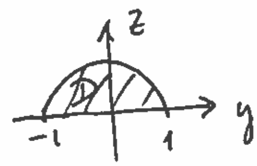
Cylindrical tunnel along x-axis

Type I region

Projection into xy-plane

$D = [-1,1] \times [-1,1]$.

$$\iint_{D_{xy}} \int_{0}^{1-y^2} x^2 e^y \, dz\, dy\, dx = \int_{-1}^{1}\int_{-1}^{1}\int_{0}^{1-y^2} x^2 e^y \, dz\, dy\, dx$$



Type II region:

Projection into yz-plane (I) $D = \{(y,z): -1 \le y \le 1,\ 0 \le z \le 1-y^2\}$

$$I = \iint_{D_{yz}} \int_{-1}^{1} x^2 e^y \, dx\, dz\, dy = \int_{-1}^{1}\int_{0}^{1-y^2}\int_{-1}^{1} x^2 e^y \, dx\, dz\, dy$$

Integration as a Type I – region

$$\iiint_E x^2 e^y \, dv = \int_{-1}^{1}\int_{-1}^{1}\int_{0}^{1-y^2} x^2 e^y \, dz\, dy\, dx =$$

$$\; x \quad y \quad z$$

$$= \int_{-1}^{1}\int_{-1}^{1} x^2 e^y (1-y^2)\, dy\, dx = \int_{-1}^{1}\int_{-1}^{1} x^2(1-y^2)e^y\, dy\, dx$$

$$= \int_{-1}^{1}\int_{-1}^{1} x^2 e^y\, dy\, dx - \int_{-1}^{1}\int_{-1}^{1} x^2 y^2 e^y\, dy\, dx =$$

$$\overset{(*)}{=} \int_{-1}^{1} x^2 (e-e^{-1})\, dx \overset{(*)}{-} \int_{-1}^{1} x^2 (2-2y+y^2) e^y \Big|_{-1}^{1} dx =$$

$$= (e-e^{-1})\frac{x^3}{3}\Big|_{-1}^{1} - \int_{-1}^{1} \left[(\cancel{2}-\cancel{2}+1)e - (2+2+1)e^{-1}\right] x^2\, dx =$$

$$= \frac{2}{3}(e-e^{-1}) - \int_{-1}^{1} e x^2\, dx + \int_{-1}^{1} 5e^{-1}x^2\, dx =$$

$$= \frac{2}{3}(\cancel{e}-e^{-1}) - \cancel{e\frac{2}{3}} + \frac{10}{3}e^{-1} = \frac{8}{3}e^{-1}$$

(*) $\displaystyle\int y^2 e^y\, dy \overset{\text{I.P.P Twice}}{=} (2-2y+y^2)\, e^y$